Uniaxial Perfectly Matched Layer Media for an Unconditionally Stable 3-D ADI–FD-TD Method

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***Abstract*—An unsplit-field perfectly matched layer (PML) medium based on Gedney's uniaxial PML (UPML) scheme is proposed for an unconditionally stable three-dimensional alternating direction implicit finite-difference time-domain (ADI–FD-TD) method. The effectiveness of the proposed ADI-UPML absorber is demonstrated through numerical example. In addition, to have a better understanding on the ADI–FD-TD method, the actual performance (i.e., while both the reflection and dispersion errors are considered) of the ADI-UPML as a function of the time step is also illustrated.**

***Index Terms*—Alternating direction implicit (ADI) method, FD-TD method, uniaxial perfectly matched layer (UPML) media.**

I. Introduction

RECENTLY, an unconditionally stable three-dimensional (3-D) alternating direction implicit finite difference-time domain (ADI–FD-TD) method has been developed [1], [2]. The main advantage of this new method is that the Courant stability condition required by the conventional FD-TD method can be totally removed. Similar to the traditional FD-TD method, one of the greatest challenges of applying the ADI–FD-TD method to open radiation problems is the development of accurate and computationally efficient absorbing boundary conditions. It has been proven that the perfectly-matched layer (PML), developed by Berenger [3], is one of the best candidates for truncating FD-TD lattices. Very recently, three different PML absorbers have been developed for the ADI–FD-TD method, and they are: the split-field PML [4] based on Berenger's original formulation, the unsplit-field PML [5] based on a ***D–H*** field formulation, and the unsplit-field CPML [6] approach. The PML [5] based on the ***D–H*** field formulation can be easily applied to arbitrary anisotropic media, whereas the CPML [6] has an advantage for low-frequency analysis.

In this letter, Gedney's uniaxial PML (UPML) absorber [7] that has been widely used for the conventional FD-TD method is extended to the ADI–FD-TD method. It is demonstrated that an ADI-UPML absorber can still be easily established with the introduction of auxiliary variables ***D*** and ***B***. Furthermore, in the proposed ADI-UPML medium, only ***D*** field updating equations need to be modified according to the principle of the ADI–FD-TD method. A potential advantage of the proposed ADI-UPML over the PMLs [5], [6] is less memory requirement. The effectiveness and the actual performance of the proposed ADI-UPML as a function of the time step are demonstrated through numerical example.

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II. Formulations for ADI-UPML Media

Similar to [7], the tensor $\hat{\varepsilon}$ and $\hat{\mu}$ used for a corner region of the ADI-UPML medium can be defined as

$$\hat{\varepsilon} = \hat{\mu} = diag\left(s_z s_y s_x^{-1},\, s_x s_z s_y^{-1},\, s_y s_x s_z^{-1}\right)$$

and

$$s_v = 1 + \frac{\sigma_v}{j\omega\varepsilon_0}, \qquad \text{where } v \text{ is } x,\, y, \text{ or } z. \tag{1}$$

As stated in [7], the convolution between the tensors and electromagnetic fields can be avoided by using the auxiliary fields ***D*** and ***B***. Hence, according to the principle of the ADI–FD-TD method [1], the updating procedure used for the x-projection of Ampere's law at time step $(n+1/2)$ and the updating procedure used for the z-projection of Faraday's law at time step $(n+1/2)$ is

$$\begin{aligned} D_x\big|_{i+1/2,\,j,\,k}^{n+1/2} &= \gamma_{y_j} D_x\big|_{i+1/2,\,j,\,k}^{n} \\ &\quad + \frac{\Delta t}{2\alpha_{y_j}^{+}}\left(\frac{H_z\big|_{i+1/2,\,j+1/2,\,k}^{n+1/2} - H_z\big|_{i+1/2,\,j-1/2,\,k}^{n+1/2}}{\Delta y}\right. \\ &\quad \left. - \frac{H_y\big|_{i+1/2,\,j,\,k+1/2}^{n} - H_y\big|_{i+1/2,\,j,\,k-1/2}^{n}}{\Delta z}\right) \end{aligned} \tag{2a}$$

$$\begin{aligned} E_x\big|_{i+1/2,\,j,\,k}^{n+1/2} &= \gamma_{z_k}\ E_x\big|_{i+1/2,\,j,\,k}^{n} + \frac{1}{\alpha_{z_k}^{+}\varepsilon_{i+1/2,\,j,\,k}} \\ &\quad \cdot\left(\alpha_{x_{i+1/2}}^{+} D_x\Big|_{i+1/2,\,j,\,k}^{n+1/2} - \alpha_{x_{i+1/2}}^{-}\ D_x\Big|_{i+1/2,\,j,\,k}^{n}\right) \end{aligned} \tag{2b}$$

$$\begin{aligned} B_z\big|_{i+1/2,\,j+1/2,\,k}^{n+1/2} &= \gamma_{x_{i+1/2}}\ B_z\big|_{i+1/2,\,j+1/2,\,k}^{n} + \frac{\Delta t}{2\alpha_{x_{i+1/2}}^{+}} \\ &\quad \cdot\left(\frac{E_x\big|_{i+1/2,\,j+1,\,k}^{n+1/2} - E_x\big|_{i+1/2,\,j,\,k}^{n+1/2}}{\Delta y}\right. \\ &\quad \left. - \frac{E_y\big|_{i+1,\,j+1/2,\,k}^{n} - E_y\big|_{i,\,j+1/2,\,k}^{n}}{\Delta x}\right) \end{aligned} \tag{3a}$$

$$H_z|^{n+1/2}_{i+1/2,\,j+1/2,\,k} = \gamma_{y_{j+1/2}}\, H_z|^n_{i+1/2,\,j+1/2,\,k} + \frac{1}{\alpha^+_{y_{j+1/2}}\mu_{i+1/2,\,j+1/2,\,k}} \cdot \left(\alpha^+_{z_k} B_z|^{n+1/2}_{i+1/2,\,j+1/2,\,k} - \alpha^-_{z_k} B_z|^n_{i+1/2,\,j+1/2,\,k}\right) \quad \text{(3b)}$$

where

$$\alpha^{\pm}_{v_w} = 1 \pm \frac{\sigma_{v_w}\Delta t}{4\varepsilon_0}$$

and

$$\gamma_{v_w} = \frac{\alpha^-_{v_w}}{\alpha^+_{v_w}}, \qquad \text{where } w = i,\ i\pm 1/2,\ j,\ j\pm 1/2,\ k \text{ or } k\pm 1/2. \quad (4)$$

From (2a), one can see that $D_x^{n+1/2}$cannot be directly updated. But, it can be updated with the help of $E_x^{n+1/2}$, $B_z^{n+1/2}$, and $H_z^{n+1/2}$ field components. Thus, by substituting these field components [i.e., (2b), (3a), and (3b)], in the order of $H_z^{n+1/2}$, $B_z^{n+1/2}$, and $E_x^{n+1/2}$, into (2a), (2a) can be rewritten as

$$\begin{aligned}
&-f_{y_{j-1}} g_{y_{j-1/2}}\, D_x|^{n+1/2}_{i+1/2,\,j-1,\,k} \\
&\quad + \left[\alpha^+_{y_j} + f_{y_j}(g_{y_{j-1/2}} + g_{y_{j+1/2}})\right] D_x|^{n+1/2}_{i+1/2,\,j,\,k} \\
&\quad - f_{y_{j+1}} g_{y_{j+1/2}}\, D_x|^{n+1/2}_{i+1/2,\,j+1,\,k} \\
&= -\gamma_{x_{i+1/2}} f_{y_{j-1}} g_{y_{j-1/2}}\, D_x|^n_{i+1/2,\,j-1,\,k} \\
&\quad + \left[\alpha^-_{y_j} + \gamma_{x_{i+1/2}} f_{y_j}(g_{y_{j-1/2}} + g_{y_{j+1/2}})\right] D_x|^n_{i+1/2,\,j,\,k} \\
&\quad - \gamma_{x_{i+1/2}} f_{y_{j+1}} g_{y_{j+1/2}}\, D_x|^n_{i+1/2,\,j+1,\,k} + \frac{\Delta t \alpha^-_{z_k}}{2\Delta y \alpha^+_{x_{i+1/2}}} \\
&\quad \cdot \Big[g_{y_{j-1/2}}\, E_x|^n_{i+1/2,\,j-1,\,k} - (g_{y_{j-1/2}} + g_{y_{j+1/2}}) \\
&\qquad \cdot E_x|^n_{i+1/2,\,j,\,k} + g_{y_{j+1/2}}\, E_x|^n_{i+1/2,\,j+1,\,k}\Big] \\
&\quad + \alpha^+_{z_k}(\gamma_{x_{i+1/2}} - \gamma_{z_k}) \\
&\quad \cdot \left(g_{y_{j+1/2}}\, B_z|^n_{i+1/2,\,j+1/2,\,k} - g_{y_{j-1/2}}\, B_z|^n_{i+1/2,\,j-1/2,\,k}\right) \\
&\quad + \frac{\Delta t}{2\Delta y}\Big(\gamma_{y_{j+1/2}}\, H_z|^n_{i+1/2,\,j+1/2,\,k} \\
&\qquad\qquad - \gamma_{y_{j-1/2}}\, H_z|^n_{i+1/2,\,j-1/2,\,k}\Big) \\
&\quad - \frac{\Delta t}{2\Delta z}\left(H_y|^n_{i+1/2,\,j,\,k+1/2} - H_y|^n_{i+1/2,\,j,\,k-1/2}\right) \\
&\quad - \frac{\Delta t \alpha^+_{z_k}}{2\Delta x \alpha^+_{x_{i+1/2}}} \\
&\quad \cdot \Big[g_{y_{j+1/2}}\left(E_y|^n_{i+1,\,j+1/2,\,k} - E_y|^n_{i,\,j+1/2,\,k}\right) \\
&\qquad - g_{y_{j-1/2}}\left(E_y|^n_{i+1,\,j-1/2,\,k} - E_y|^n_{i,\,j-1/2,\,k}\right)\Big]
\end{aligned} \quad \text{(5a)}$$

where

$$f_{y_j} = \frac{\Delta t}{2\Delta y \varepsilon_{i+1/2,\,j,\,k}} \quad \text{and} \quad g_{y_j} = \frac{\Delta t}{2\Delta y \mu_{i+1/2,\,j,\,k}\alpha^+_{y_j}}. \quad \text{(5b)}$$

Equation (5) is the final updating equation of D_x at time step $(n+1/2)$. Similarly, one can also obtain the final updating equation of D_x at time step $(n+1)$. Once D_x, D_y, and D_z are updated with the manner described in this letter, the rest of the field components can be updated with a similar way to the conventional UPML scheme. Moreover, (5) results in a tridiagonal matrix system, and it can be easily solved with the approach presented in [8].

III. Numerical Validations

To validate the ADI-UPML absorber, a point source radiating in free-space is studied. A uniform mesh with $\Delta x = \Delta y = \Delta z = 1.0$ mm is used. The thickness of the PML layers is $N = 10$ cells, and the total mesh dimension is 30 × 30 × 31. The E_z field component at the center of the domain is used for the excitation. The time dependence of the source is a differentiated Gaussian pulse with a half-bandwidth of 3.125 GHz. To ensure the numerical results are accurate enough for all the field components, the excitation function is applied to both the first and second sub-iteration [8]. Within the ADI-UPML, the coefficients (i.e., $\alpha_v^{\pm}$ and γ_v) used in the updating equations are chosen similarly to [9]. For example, if an auxiliary parameter, vn, is defined as

$$vn(p) = a * \left(\frac{p}{N}\right)^m, \qquad p = 1, 2, \ldots, N \quad (6)$$

where a is a parameter used to control the profile of vn, then the coefficient $\alpha_v^{\pm}$ and γ_v can be expressed

$$\alpha_v^{\pm} = 1 \pm vn(p) \quad \text{and} \quad \gamma_v = \frac{1 - vn(p)}{1 + vn(p)}. \quad (7)$$

In this way, one does not need to vary the conductivity profile. Also, the time step Δt is implicitly used in the coefficients and thus it prevents rapid change (which might affect the performance of PML) of the coefficients when a very big time step is used. A number of simulations were done to find the optimal value of a, and it was found that if $a = 0.4$ then PML gives the best performance for $m = 4$ and $N = 10$. The reflection error of the ADI-UPML is studied by analyzing the relative error of the E_z field component recorded at point A (which is in the same plane of the source and one cell from the PML). When the time step Δt increases, two important error sources are involved in the ADI–FD-TD method: the reflection error caused by the ADI-UPML and the dispersion error of the ADI–FD-TD method itself. Therefore, to have a better understanding on the performance of the ADI–FD-TD method, both the reflection error and the dispersion error should be considered. As stated in [4], [6], the effectiveness of the ADI-UPML can be evaluated when only the reflection error is considered. But, to evaluate the actual performance of the ADI-UPML, both the reflection and dispersion errors have to be considered. First, to evaluate the effectiveness of the ADI-UPML, a reference solution based on an extended lattice is computed for each CFLN (where CFLN $= \Delta t/\Delta t_{\max}^{FD-TD}$, and in our case $\Delta t_{\max}^{FD-TD} = 1.926$ ps). Fig. 1 shows the reflection error of the proposed ADI-UPML as a function of CFLN at point A. For comparison, the reflection error of the conventional UPML (with CFLN = 1.0) is also plotted in Fig. 1. The results in Fig. 1 indicate that the reflection error of the ADI-UPML is still at an acceptable level

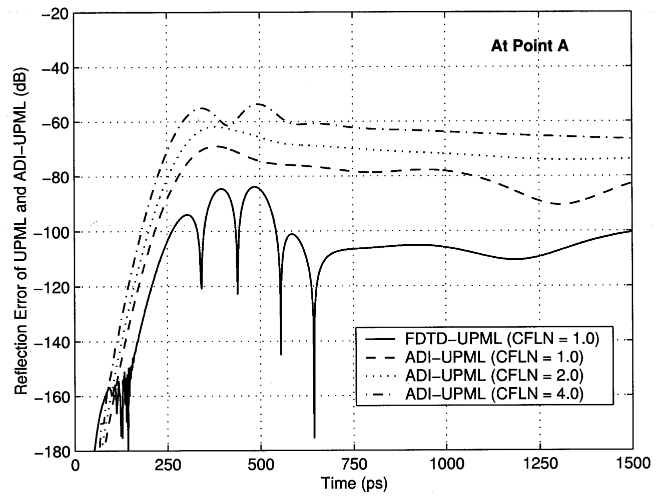

Fig. 1. Relative reflection error of the ADI-UPML as a function of time steps.

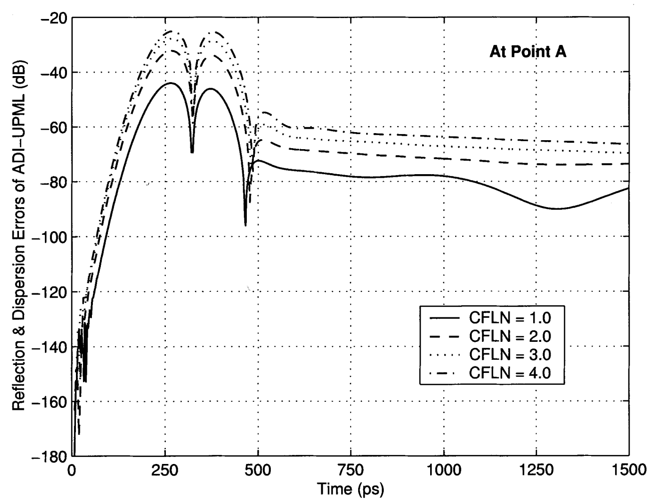

Fig. 2. The actual performance of the ADI-UPML.

(i.e., around -40 dB) when CFLN increases to 4. However, it must be pointed out that this only confirms the effectiveness of the ADI-UPML. The actual performance (referred to the reference solution of the standard FD-TD method with CFLN $= 1.0$) of the ADI-UPML at point A as a function of CFLN is shown in Fig. 2. The level of the dispersion error induced by the ADI–FD-TD method can be estimated by comparing the results in Figs. 1 and 2. Moreover, it can be seen from Fig. 2 that the accuracy of the numerical results obtained with CFLN $= 4.0$ will no longer be acceptable due to the stronger dispersion error of the ADI–FD-TD method. This just substantiates that a comprehensively study on the numerical dispersion property of the ADI–FD-TD method [10] is important.

IV. Conclusions

In this letter, the unsplit-field perfectly-matched layer based on Gedney's UPML absorbers is developed for the 3-D ADI–FD-TD method. Similar to the conventional UPML, in the proposed ADI-UPML, the convolution between the absorbing tensors and electromagnetic fields is avoided by introducing two auxiliary $\boldsymbol{D}$ and $\boldsymbol{B}$ variables. It is demonstrated that, with the theory proposed in this letter, the updating procedure for the field components (except for the $\boldsymbol{D}$ field components) used for the ADI-UPML is very similar to those used for the conventional UPML scheme. The ADI–FD-TD method implemented with the proposed ADI-UPML remains stable beyond the Courant stability limit of the conventional FD-TD method.

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